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Advanced Linear Algebra (MA 409)
Problem Sheet - 13

Systems of Linear Equations – Theoretical Aspects

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1. Label the following statements as true or false.
   (a) Any system of linear equations has at least one solution.
   (b) Any system of linear equations has at most one solution.
   (c) Any homogeneous system of linear equations has at least one solution.
   (d) Any system of n linear equations in n unknowns has at most one solution.
   (e) Any system of n linear equations in n unknowns has at least one solution.
   (f) If the homogeneous system corresponding to a given system of linear equations has a solution, then the given system has a solution.
   (g) If the coefficient matrix of a homogeneous system of n linear equations in n unknowns is invertible, then the system has no nonzero solutions.
   (h) The solution set of any system of m linear equations in n unknowns is a subspace of F^n.

2. For each of the following homogeneous systems of linear equations, find the dimension of and a basis for the solution set.

   a) $x_1 + 3x_2 = 0$
   $2x_1 + 6x_2 = 0$

   b) $x_1 + x_2 - x_3 = 0$
   $4x_1 + x_2 - 2x_3 = 0$

   c) $x_1 + 2x_2 - x_3 = 0$
   $2x_1 + x_2 + x_3 = 0$

   d) $2x_1 + x_2 - x_3 = 0$
   $x_1 - x_2 + x_3 = 0$
   $x_1 + 2x_2 - 2x_3 = 0$

   e) $x_1 + 2x_2 - 3x_3 + x_4 = 0$

   f) $x_1 + 2x_2 = 0$
   $x_1 - x_2 = 0$

   g) $x_1 + 2x_2 + x_3 + x_4 = 0$
   $x_2 - x_3 + x_4 = 0$

3. Using the results of Exercise 2, find all solutions to the following systems.

   a) $x_1 + 3x_2 = 5$
   $2x_1 + 6x_2 = 10$

   b) $x_1 + x_2 - x_3 = 1$
   $4x_1 + x_2 - 2x_3 = 3$

   c) $x_1 + 2x_2 - x_3 = 3$
   $2x_1 + x_2 + x_3 = 6$

   d) $2x_1 + x_2 - x_3 = 5$
   $x_1 - x_2 + x_3 = 1$
   $x_1 + 2x_2 - 2x_3 = 4$

   e) $x_1 + 2x_2 - 3x_3 + x_4 = 1$

   f) $x_1 + 2x_2 = 5$
   $x_1 - x_2 = -1$

   g) $x_1 + 2x_2 + x_3 + x_4 = 1$
   $x_2 - x_3 + x_4 = 1$

4. For each system of linear equations with the invertible coefficient matrix A,

   (1) Compute A^{-1}.

   (2) Use A^{-1} to solve the system.

   (a) $x_1 + 3x_2 = 4$
   $2x_1 + 5x_2 = 3$

   (b) $x_1 + 2x_2 - x_3 = 5$
   $x_1 + x_2 + x_3 = 1$
   $2x_1 - 2x_2 + x_3 = 4$

5. Give an example of a system of n linear equations in n unknowns with infinitely many solutions.

6. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $T(a, b, c) = (a + b, 2a - c)$. Determine $T^{-1}(1, 11)$.

7. Determine which of the following systems of linear equations has a solution.

   a) $x_1 + x_2 - x_3 + 2x_4 = 2$
   $x_1 + x_2 + 2x_3 = 1$
   $2x_1 + 2x_2 + x_3 + 2x_4 = 4$

   b) $x_1 + x_2 - x_3 = 1$
   $2x_1 + x_2 + 3x_3 = 2$

   c) $x_1 + 2x_2 + 3x_3 = 1$
   $x_1 + x_2 - x_3 = 0$
   $x_1 + 2x_2 + x_3 = 3$

   d) $x_1 + x_2 + 3x_3 - x_4 = 0$
   $x_1 + x_2 + x_3 + x_4 = 1$
   $x_1 - 2x_2 + x_3 - x_4 = 1$
   $4x_1 + x_2 + 8x_3 - x_4 = 0$

   e) $x_1 + 2x_2 - x_3 = 1$
   $2x_1 + x_2 + 2x_3 = 3$
   $x_1 - 4x_2 + 7x_3 = 4$

8. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(a, b, c) = (a + b, b - 2c, a + 2c)$. For each vector v in $\mathbb{R}^3$, determine whether $v \in R(T)$.
   (a) $v = (1, 3, -2)$ (b) $v = (2, 1, 1)$

9. Prove that the system of linear equations $Ax = b$ has a solution if and only if $b \in R(L_A)$.

10. Prove or give a counterexample to the following statement: If the coefficient matrix of a system of m linear equations in n unknowns has rank m, then the system has a solution.

11. In the closed model of Leontief with food, clothing, and housing as the basic industries, suppose that the input-output matrix is

$$A = \begin{pmatrix} \frac{7}{16} & \frac{1}{2} & \frac{3}{16} \\ \frac{5}{16} & \frac{1}{6} & \frac{5}{16} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{2} \end{pmatrix}.$$

    At what ratio must the farmer, tailor, and carpenter produce in order for equilibrium to be attained?

12. A certain economy consists of two sectors: goods and services. Suppose that 60% of all goods and 30% of all services are used in the production of goods. What proportion of the total economic output is used in the production of goods?

13. In the notation of the open model of Leontief, suppose that

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{5} \end{pmatrix} \quad \text{and} \quad d = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

are the input-output matrix and the demand vector, respectively. How much of each commodity must be produced to satisfy this demand?

14. A certain economy consisting of the two sectors of goods and services supports a defense system that consumes \$90 billion worth of goods and \$20 billion worth of services from the economy but does not contribute to economic production. Suppose that 50 cents worth of goods and 20 cents worth of services are required to produce \$1 worth of goods and that 30 cents worth of goods and 60 cents worth of services are required to produce \$1 worth of services. What must the total output of the economic system be to support this defense system?

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